New Mixing-Length Model for Turbulent High-Speed Flows

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Abstract

THE new mixing-length model developed here to account for compressibility on turbulence for high-speed flows is a generalization of Prandtl's original mixing-length model. It involves a new parameter S, which acts as an effective turbulent Schmidt number for mixtures or a turbulent Prandtl number for a homogeneous gas. The new model and Prandtl's original model are both applied to six test cases including high Mach number flows over a flat surface, tangential slot injection, and shock/turbulent shear-layer interactions. The numerical method of flux splitting with Roe's scheme to solve the time-averaged Reynolds form of the mean, thin-layer, Navier-Stokes equations has been adopted. The predictions with the new mixing-length model are generally in better agreement with the data in cases where there is a large difference between the two predictions.

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One of the important Reynolds stresses for thin viscous layers is

$$\tau'_{xy} = -\bar{\rho}\,\overline{u'v'} - \bar{u}\,\overline{\rho'v'} - \bar{v}\,\overline{\rho'u'} - \overline{\rho'u'v'} \tag{1}$$

In thin layers, $\bar{u} \gg \bar{v}$, so $\bar{u}\,\overline{\rho'v'} \gg \bar{v}\,\overline{\rho'u'}$. Also, the term $\overline{\rho'u'v'}$ is higher order. Thus,

$$\tau'_{xy} \approx -\bar{\rho}\,\overline{u'v'} - \bar{u}\,\overline{\rho'v'} \tag{2}$$

Generally, one can write

$$\overline{u'v'} = -c_{uv}\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}} \tag{3}$$

Assuming $u' \sim \partial\bar{u}/\partial y$ and $\sqrt{\overline{v'^2}} \approx \sqrt{\overline{u'^2}}$, following Prandtl,

$$\sqrt{\overline{u'^2}} = l_u \frac{\partial \bar{u}}{\partial y} \tag{4a}$$

$$\sqrt{\overline{v'^2}} = \text{const}\sqrt{\overline{u'^2}} = \text{const}\; l_u \frac{\partial \bar{u}}{\partial y} \tag{4b}$$

From the preceding relations,

$$-\bar{\rho}\,\overline{u'v'} = \bar{\rho}(c_{uv}\cdot\text{const}\, l_u^2)\frac{\partial \bar{u}}{\partial y}\frac{\partial \bar{u}}{\partial y} = \bar{\rho} l_m^2 \frac{\partial \bar{u}}{\partial y}\frac{\partial \bar{u}}{\partial y} \tag{5}$$

This is Prandtl's original mixing-length hypothesis.[1]

Assuming now $\sqrt{\overline{\rho'^2}} = l_\rho \dfrac{\partial \bar{\rho}}{\partial y}$, one obtains

$$\begin{aligned}\overline{\rho'v'} &= -c_{\rho v}\sqrt{\overline{\rho'^2}}\sqrt{\overline{v'^2}} \\ &= -c_{\rho v} l_\rho \cdot \text{const}\, l_u \frac{\partial \bar{\rho}}{\partial y}\frac{\partial \bar{u}}{\partial y} = -\frac{l_m^2}{S}\frac{\partial \bar{\rho}}{\partial y}\frac{\partial \bar{u}}{\partial y}\end{aligned}$$

where $S = (c_{uv} l_u)/(c_{\rho v} l_\rho)$. Finally,

$$-\bar{u}\,\overline{\rho'v'} = l_m^2 \frac{\bar{u}}{S}\frac{\partial \bar{\rho}}{\partial y}\frac{\partial \bar{u}}{\partial y} \tag{6}$$

Putting Eqs. (5) and (6) into Eq. (2), we obtain

$$\tau'_{xy} = l_m^2\left(\bar{\rho}\frac{\partial \bar{u}}{\partial y} + \frac{\bar{u}}{S}\frac{\partial \bar{\rho}}{\partial y}\right)\frac{\partial \bar{u}}{\partial y} \tag{7}$$

For physical reasons, one needs an absolute value sign:

$$\tau'_{xy} = l_m^2\left|\bar{\rho}\frac{\partial \bar{u}}{\partial y} + \frac{\bar{u}}{S}\frac{\partial \bar{\rho}}{\partial y}\right|\frac{\partial \bar{u}}{\partial y} \tag{8}$$

For thin layers, $\mu_t = \tau'_{xy}/(\partial\bar{u}/\partial y)$, so

$$\mu_t = l_m^2\left|\bar{\rho}\frac{\partial \bar{u}}{\partial y} + \frac{\bar{u}}{S}\frac{\partial \bar{\rho}}{\partial y}\right| \tag{9}$$

From gradient transport theory for a mixture

$$\overline{\rho'v'} = -D_t\frac{\partial \bar{\rho}}{\partial y} \tag{10}$$

With the turbulent Schmidt number Sc_t,

$$\bar{\rho}D_t = \frac{\mu_t}{Sc_t} \approx l_m^2 \frac{\bar{\rho}}{Sc_t}\frac{\partial \bar{u}}{\partial y} \tag{11}$$

where we use, for convenience, the Prandtl formula. Thus,

$$-\bar{u}\,\overline{\rho'v'} = \bar{u}D_t\frac{\partial \bar{\rho}}{\partial y} \approx l_m^2 \frac{\bar{u}}{Sc_t}\frac{\partial \bar{\rho}}{\partial y}\frac{\partial \bar{u}}{\partial y} \tag{12}$$

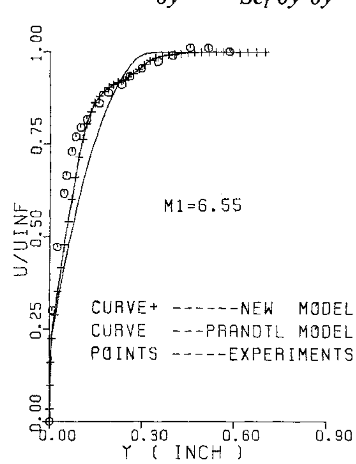

Fig. 1 Comparison of predictions and data from Ref. 2 for $M = 6.55$ flat plate flow.

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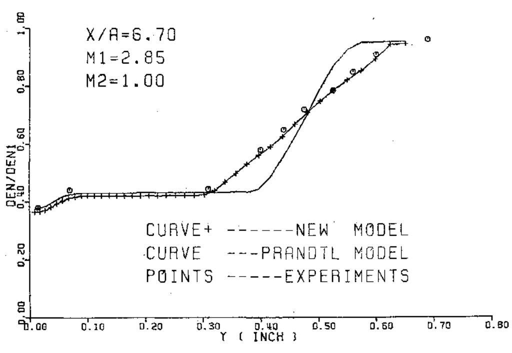
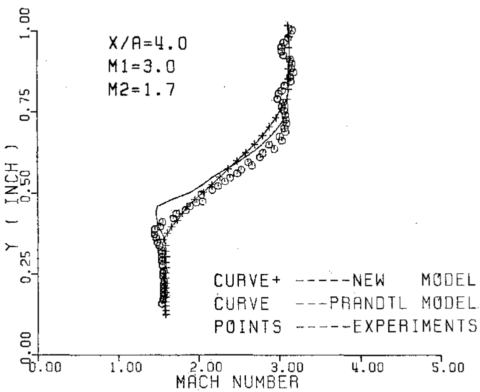
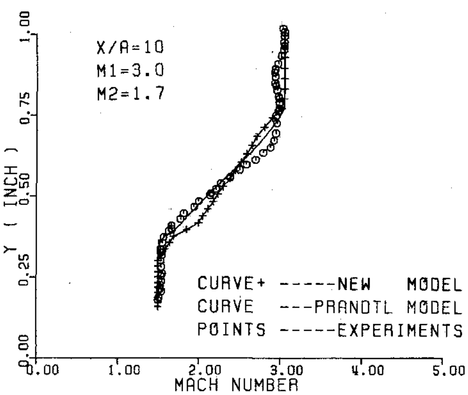


x/a = 6.70

Fig. 2 Comparison of predictions and data from Ref. 3 for slot injection with $M_1 = 2.85$ and $M_2 = 1.00$.



a) $x/a = 4.0$



b) $x/a = 10.0$

Fig. 3 Comparison of predictions and data from Ref. 4 for heated slot injection with $M_1 = 3.0$ and $M_2 = 1.7$.

Comparing Eq. (6) with Eq. (12), it can be seen that S is, approximately, the turbulent Schmidt number. For a single species gas, we cannot use the concept of a Schmidt number.

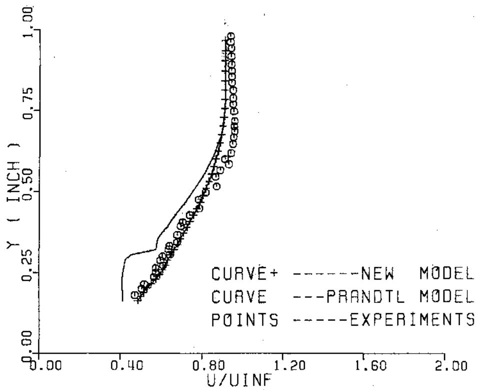


Fig. 4 Comparison of predictions and data from Ref. 4 for profiles after shock/heated shear-layer interaction.

If the turbulent Lewis number is unity, the energy equation for a multispecies gas will reduce to the energy equation for a single species gas. So, one can consider that the single species case corresponds to a Lewis number equal to one and the value of S is close to Pr_t. We have adopted $S \approx Pr_t \approx Sc_t \approx 0.8$.

Results and Discussion

To test the new model, six cases were chosen, and solutions with both the current model and Prandtl's model were obtained and compared with experimental results.

The first test cases were high-speed turbulent boundary layers over a flat plate with heat transfer at $M_1 = 4.67$ and $M_1 = 6.55$.

The second cases are sonic and supersonic slot injection flows. The third slot injection case had a thick initial boundary.

The next case was a heated, supersonic slot injection problem, with $u_2/u_1 = 0.86$.

The final test cases involve shock/turbulent shear-layer interactions.

A few typical results from this wide range of test cases are shown in Figs. 1–4. For cases with a large difference between predictions, the new model is generally better. Thus, we conclude that the new model includes more of the physics of turbulent flows with strong compressibility effects.

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